

# SETS OF INDEPENDENT POSTULATES FOR BETWEENNESS\*

BY

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## INTRODUCTION

The "universe of discourse" of the present paper is the class of all well-defined systems  $(K, R)$  where  $K$  is any class of elements  $A, B, C, \dots$ , and  $R$  is any triadic relation. The notation  $R[ABC]$ , or simply  $ABC$ , indicates that three given elements  $A, B, C$ , in the order stated, satisfy the relation  $R$ .

Examples of such systems  $(K, R)$  are the following, of which example (a) is the most important:

(a)  $K$  is the class of points on a line;  $AXB$  means that the point  $X$  lies between the points  $A$  and  $B$ .

(b)  $K$  is the class of natural numbers;  $AXB$  means that the number  $X$  is the product of the numbers  $A$  and  $B$ .

(c)  $K$  is the class of human beings;  $AXB$  means that  $X$  is a descendant of  $A$  and an ancestor of  $B$ .

(d)  $K$  is the class of points on the circumference of a circle;  $AXB$  means that the arc  $A-X-B$  is less than  $180^\circ$ .

(e)  $K$  is a class comprising four elements, namely, the numbers 2, 6,  $-6$ , and 648;  $AXB$  means  $X^4 = A \times B$ .

It is obvious that these systems, and others like them, will possess a great variety of properties expressible in terms of the fundamental variables  $K$  and  $R$ . The object of this paper is to state clearly the characteristic properties of the type of system represented by example (a) above, by which this type of system is distinguished from all other possible systems  $(K, R)$ .

In Section 1, we give a basic list of twelve postulates, due essentially to Pasch,† from which various sets of *independent* postulates will later be selected.

\* Presented to the Society, September 5, 1916. The parts of the paper which do not involve the postulates here numbered 5 and 8 were presented by Professor Huntington at the meetings of December 31, 1912, and April 26, 1913. The necessity of adding postulates 5 and 8 was kindly pointed out by Professor R. L. Moore, and all the theorems and examples which involve these two postulates are due to Dr. Kline.

† M. Pasch, *Vorlesungen über neuere Geometrie*, Leipzig, 1882; G. Peano, *Sui fondamenti della geometria*, *Rivista di Matematica*, vol. 4 (1894), pp. 51-90; F. Schur, *Grundlagen der Geometrie*, Leipzig, 1909. Other sets of postulates for betweenness have been given

These postulates are all “general laws” as distinguished from “existence postulates,” and include, in fact, all the possible general laws of linear order concerning not more than four elements.

In Sections 2 and 3 we give an exhaustive discussion of all the possible ways in which any one of these basic postulates can be deduced from any others of the list, and in Section 4, we give an exhaustive list of all the distinct sets of independent postulates (eleven in number) which can be selected from the basic list.\*

Any one of these sets of independent postulates may be used, as in Section 5, to define the type of system  $(K, R)$  which we are considering—that is, to define the relation of betweenness.

The existence postulates which might be imposed, in addition to the general laws, would serve to distinguish the various sub-types which are included within the general type of system  $(K, R)$  here considered. These existence postulates, such as the postulates of discreteness, density, continuity, etc., are already well known, and will not be discussed further in the present paper.†

## 1. BASIC LIST OF TWELVE POSTULATES

In this section we give the basic list of twelve postulates from which various sets of independent postulates will later be selected.

The first four postulates, A–D, concern three elements.

POSTULATE A.  $AXB \supset BXA$ .

That is, if  $AXB$  is true, then  $BXA$  is true. In other words, in the notation  $ABC$ , an interchange of the terminal elements is always allowable.

POSTULATE B.

$A \neq B \cdot B \neq C \cdot C \neq A \supset BAC \sim CAB \sim ABC \sim CBA \sim ACB \sim BCA$ .

That is, if  $A, B, C$ , are distinct, then *at least one* of the three elements will occupy the middle position in a true triad.

POSTULATE C.  $A \neq X \cdot X \neq Y \cdot Y \neq A \supset AXY \cdot AYX = \cdot 0$ .

by D. Hilbert; *Grundlagen der Geometrie*, 1899, third edition 1909; and by O. Veblen: *A system of axioms for geometry*, these *Transactions*, vol. 5 (1904), pp. 343–384, or *The foundations of geometry*, in the volume called *Monographs on Topics of Modern Mathematics*, edited by J. W. A. Young, 1911, pp. 1–51.

\* The postulates of each of these sets are independent of each other in the ordinary sense of the term “independence”; that is, no postulate of any one set can be deduced from the remaining postulates of that set. It is probable that the postulates of each set are also “completely independent” in the sense suggested by E. H. Moore in his *Introduction to a Form of General Analysis* (*New Haven Colloquium*, 1906, published by the Yale University Press, New Haven, 1910, p. 82); but no attempt to discuss the “complete existential theory” of the postulates, in the sense there defined, has here been made.

† See, for example, E. V. Huntington, *The continuum as a type of order*, reprinted from the *Annals of Mathematics*, 1905 (Publication Office of Harvard University); second edition, Harvard University Press, 1917.

That is, if  $A, X, Y$ , are distinct, we cannot have  $AXY$  and  $AYX$  both true at the same time.

From Postulates A and C it follows that if  $A, B, C$  are distinct elements, then *not more than one* of the three elements can occupy the middle position in a true triad.

From Postulates A, B, and C, together, it follows that if  $A, B$ , and  $C$  are distinct elements, then one and only one of the triads  $ABC, BCA, CAB$  will be true.

POSTULATE D.  $ABC : \supset A \neq B . B \neq C . C \neq A$ .

That is, if  $ABC$  is true, then the elements  $A, B$ , and  $C$ , are distinct.

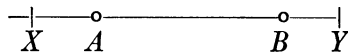
The remaining eight postulates are concerned with four distinct elements.

POSTULATES 1-8. If

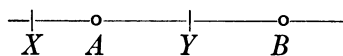
$$A \neq B . A \neq X . A \neq Y . B \neq X . B \neq Y . X \neq Y,$$

then

$$1. XAB . ABY . \supset XAY;$$

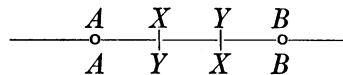


$$2. XAB . AYB . \supset XAY;$$



$$3. XAB . AYB . \supset XYB;$$

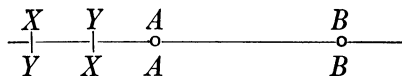
$$4. AXB . AYB . \supset AXY \sim AYX;$$



$$5. AXB . AYB . \supset AXY \sim YXB;$$

$$6. XAB . YAB . \supset XYB \sim YXB;$$

$$7. XAB . YAB . \supset XYA \sim YXA;$$



$$8. XAB . YAB . \supset XYA \sim YXB.$$

The eight Postulates 1-8 (together with the analogous postulates obtained from these by the aid of Postulate A alone) include *all the possible "general laws" of betweenness concerning four distinct elements*. For, if we think of  $A$  and  $B$  as two given points on a line, the hypotheses of these postulates state all the possible relations in which two other distinct points  $X$  and  $Y$  of the line can stand in regard to  $A$  and  $B$ . (See, however, the Appendix.)

We shall see later that no further general laws—that is, no general laws concerning more than four distinct elements—need be assumed as fundamental. (Existence postulates, which play a very different rôle from the general laws, are not here considered.)

## 2. THEOREMS ON DEDUCIBILITY

In this section we take up all the cases in which the following question is to be answered in the affirmative:

*Given, any subset  $S$  of the twelve postulates of our basic list, and any postulate  $P$  of the list, not belonging to  $S$ ; is  $P$  deducible from  $S$ ?*

The answers are comprised in the following 71 theorems. In the proofs of these theorems, all the steps are given explicitly, except those depending only on Postulate A. Moreover, in case any postulate (except Postulate A) is used more than once in a proof, the frequency of its use is indicated by an exponent; this latter information, however, is added merely as a matter of possible interest to the reader, and the omission of the exponents would not affect the conclusions of the paper in any way.\*

A summary of the theorems will be found at the end of § 2.

#### PROOFS OF POSTULATE 1

THEOREM 1a. *Proof of 1 from A, B, C<sup>3</sup>, 2, 4.*

To prove:  $XAB \cdot ABY \cdot \supset \cdot XAY$ . By B,  $XAY \sim AYX \sim AXY$ . Suppose  $AXY$ . Then by 4,  $ABY \cdot AXY \cdot \supset \cdot ABX \sim AXB$ , contrary to  $XAB$ , by C. Suppose  $AYX$ . Then by 2,  $BAX \cdot AYX \cdot \supset \cdot BAY$ , contrary to  $ABY$ , by C. Therefore  $XAY$ .

THEOREM 1b. *Proof of 1 from A, B, C<sup>2</sup>, 3, 4.*

To prove:  $XAB \cdot ABY \cdot \supset \cdot XAY$ . By C,  $AXB$  and  $ABX$  are false, since  $XAB$  is true. By B,  $XYA \sim AXY \sim XAY$ .

Case 1. Suppose  $XYA$ . Then by 3,  $XYA \cdot YBA \cdot \supset \cdot XBA$ , which is false.

Case 2. Suppose  $AXY$ . Then by 4,  $AXY \cdot ABY \cdot \supset \cdot AXB \sim ABX$ , which are both false. Therefore  $XAY$ .

THEOREM 1c. *Proof of 1 from A, B, C<sup>3</sup>, 2<sup>2</sup>, 5.*

To prove:  $XAB \cdot ABY \cdot \supset \cdot XAY$ . By B,  $XAY \sim AYX \sim AXY$ . Suppose  $AXY$ . Then by 5,  $AXY \cdot ABY \cdot \supset \cdot AXB \sim BXY$ . But  $AXB$  is contrary to  $XAB$ , by C; while if  $BXY$ , then by 2,  $ABY \cdot BXY \cdot \supset \cdot ABX$ , contrary to  $XAB$ , by C.

Suppose  $AYX$ . Then by 2,  $BAX \cdot AYX \cdot \supset \cdot BAY$ , contrary to  $ABY$ , by C. Therefore  $XAY$ .

THEOREM 1d. *Proof of 1 from A, B, C, 3<sup>2</sup>, 5.*

To prove:  $XAB \cdot ABY \cdot \supset \cdot XAY$ . By C,  $XBA$  is false, since  $XAB$  is true. By B,  $XYA \sim AXY \sim XAY$ .

Case 1. Suppose  $XYA$ . Then by 3,  $XYA \cdot YBA \cdot \supset \cdot XBA$ , which is false.

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\* We are indebted to Mr. R. M. Foster, of Harvard University, for reductions in the "frequency exponents" (chiefly in regard to Postulate C) in the following theorems: 1b, 1d; 2c, 2g; 3a, 3b; 4b; 5f; 6j; 7b, 7c, 7j; 8b, 8c, 8d, 8f, 8j. A notion similar to that of "frequency exponents" was introduced by H. Brandes in his Halle Dissertation, 1908, *Über die axiomatische Einfachheit, mit besonderer Berücksichtigung der auf Addition beruhenden Zerlegungsweise des Pythagoräischen Lehrsatzes*. Compare F. Bernstein, *Ueber die axiomatische Einfachheit von Beweisen*, Atti del IV Congresso Internazionale dei Matematici, Roma, 1908, vol. 3 (1909), pp. 391-392, and also E. Lemoine's *Géométrie* of 1902.

*Case 2.* If  $AXY$ , then by 5,  $ABY \cdot AXY \cdot \supset \cdot ABX \sim XBY$ ; but  $ABX$  is false, and if  $XBY$ , then by 3,  $YBX \cdot BAX \cdot \supset \cdot YAX$ . Therefore  $XAY$ .

### PROOFS OF POSTULATE 2

**THEOREM 2a.** *Proof of 2 from A, B, C<sup>3</sup>, 1<sup>3</sup>, 7.*

To prove:  $XAB \cdot AYB \cdot \supset \cdot XAY$ . By B,  $XAY \sim AYX \sim AXY$ . Suppose  $AYX$ . Then by 7,  $BYA \cdot XYA \cdot \supset \cdot BXY \sim XBY$ . But if  $BXY$ , then by 1,  $BXY \cdot XYA \cdot \supset \cdot BXA$ , contrary to  $XAB$ , by C; and if  $XBY$ , then by 1,  $XBY \cdot BYA \cdot \supset \cdot XBA$ , contrary to  $XAB$ , by C. Suppose  $AXY$ . Then by 1,  $BAX \cdot AXY \cdot \supset \cdot BAY$ , contrary to  $AYB$ , by C. Therefore  $XAY$ .

**THEOREM 2b.** *Proof of 2 from A, B, C<sup>3</sup>, 1, 6.*

To prove:  $XAB \cdot AYB \cdot \supset \cdot XAY$ . By B,  $XAY \sim AYX \sim YXA$ . Suppose  $YXA$ . Then by 1,  $BAX \cdot AXY \cdot \supset \cdot BAY$ , contrary to  $AYB$ , by C. Suppose  $AYX$ . Then by 6,  $XYA \cdot BYA \cdot \supset \cdot XBA \sim BXA$ , contrary to  $XAB$ , by C. Therefore  $XAY$ .

**THEOREM 2c.** *Proof of 2 from A, B, C<sup>2</sup>, 3, 6.*

To prove:  $XAB \cdot AYB \cdot \supset \cdot XAY$ . By C,  $BXA$  and  $XBA$  are false, since  $XAB$  is true. By B,  $YXA \sim XYA \sim XAY$ .

*Case 1.* Suppose  $YXA$ . Then by 3,  $BYA \cdot YXA \cdot \supset \cdot BXA$ , which is false.

*Case 2.* Suppose  $XYA$ . Then by 6,  $BYA \cdot XYA \cdot \supset \cdot BXA \sim XBA$ , which are both false. Therefore  $XAY$ .

**THEOREM 2d.** *Proof of 2 from A, C, 3<sup>2</sup>, 7.*

To prove:  $XAB \cdot AYB \cdot \supset \cdot XAY$ . By 3,  $XAB \cdot AYB \cdot \supset \cdot XYB$ . Hence by 7,  $XYB \cdot AYB \cdot \supset \cdot XAY \sim AXY$ . But if  $AXY$ , then by 3,  $BYA \cdot YXA \cdot \supset \cdot BXA$ , contrary to  $XAB$ , by C. Therefore  $XAY$ .

**THEOREM 2e.** *Proof of 2 from A, C<sup>2</sup>, 3, 4, 6.*

To prove:  $XAB \cdot AYB \cdot \supset \cdot XAY$ . By 3,  $XAB \cdot AYB \cdot \supset \cdot XYB$ . Hence by 4,  $XYB \cdot XAB \cdot \supset \cdot XYA \sim XAY$ . But if  $XYA$ , then by 6,  $XYA \cdot BYA \cdot \supset \cdot XBA \sim BXA$ , contrary to  $XAB$ , by C. Therefore  $XAY$ .

**THEOREM 2f.** *Proof of 2 from A, B, C<sup>3</sup>, 1<sup>2</sup>, 8.*

To prove:  $XAB \cdot AYB \cdot \supset \cdot XAY$ . By B,  $XAY \sim AYX \sim YXA$ . Suppose  $YXA$ . Then by 1,  $BAX \cdot AXY \cdot \supset \cdot BAY$ , contrary to  $AYB$ , by C. Suppose  $AYX$ . Then by 8,  $XYA \cdot BYA \cdot \supset \cdot XBY \sim BXA$ . But  $BXA$  is contrary to  $XAB$ , by C; while if  $XBY$ , then by 1,  $XBY \cdot BYA \cdot \supset \cdot XBA$ , contrary to  $XAB$ , by C. Therefore  $XAY$ .

**THEOREM 2g.** *Proof of 2 from A, B<sup>2</sup>, C<sup>3</sup>, 1<sup>3</sup>, 5.*

To prove:  $XAB \cdot AYB \cdot \supset \cdot XAY$ . By C,  $BAY$  and  $XBA$  and  $BXA$

are false, since  $AYB$  and  $XAB$  are true. By B,  $AXY \sim XYA \sim XAY$ ; and by B,  $XBY \sim BXY \sim BYX$ .

*Case 1.* Suppose  $AXY$ . Then by 1,  $BAX \cdot AXY \cdot \supset \cdot BAY$ , which is false.

*Case 2.* Suppose  $XBY$ . Then by 1,  $XBY \cdot BYA \cdot \supset \cdot XBA$ , which is false.

*Case 3.* Suppose  $XYA$  and  $BXY$ . Then by 1,  $BXY \cdot XYA \cdot \supset \cdot BXA$ , which is false.

*Case 4.* Suppose  $BYX$ . Then by 5,  $BAX \cdot BYX \cdot \supset \cdot BAY \sim YAX$ , where  $BAY$  is false. Therefore  $XAY$ .

THEOREM 2h. *Proof of 2 from A, C, 3, 8.*

To prove:  $XAB \cdot AYB \cdot \supset \cdot XAY$ . By 3,  $XAB \cdot AYB \cdot \supset \cdot XYB$ . Hence by 8,  $XYB \cdot AYB \cdot \supset \cdot XAY \sim AXB$ . But  $AXB$  is contrary to  $XAB$ , by C. Therefore  $XAY$ .

THEOREM 2i. *Proof of 2 from A, C, 3, 5.*

To prove:  $XAB \cdot AYB \cdot \supset \cdot XAY$ . By 3,  $XAB \cdot AYB \cdot \supset \cdot XYB$ . Hence by 5,  $XAB \cdot XYB \cdot \supset \cdot XAY \sim YAB$ . But  $YAB$  is contrary to  $AYB$ , by C. Therefore  $XAY$ .

### PROOFS OF POSTULATE 3

THEOREM 3a. *Proof of 3 from A, B<sup>2</sup>, C<sup>3</sup>, I<sup>4</sup>.*

To prove:  $XAB \cdot AYB \cdot \supset \cdot XYB$ . By C,  $XBA$  and  $BAY$  and  $BXA$  are false, since  $XAB$  and  $AYB$  are true. By B,  $XBY \sim BXY \sim XYB$ ; and by B,  $AXY \sim XYA \sim YAX$ .

*Case 1.* Suppose  $XBY$ . Then by 1,  $XBY \cdot BYA \cdot \supset \cdot XBA$ , which is false.

*Case 2.* Suppose  $AXY$ . Then by 1,  $BAX \cdot AXY \cdot \supset \cdot BAY$ , which is false.

*Case 3.* Suppose  $BXY$  and  $XYA$ . Then by 1,  $BXY \cdot XYA \cdot \supset \cdot BXA$ , which is false.

*Case 4.* Suppose  $YAX$ . Then by 1,  $BYA \cdot YAX \cdot \supset \cdot BYX$ . Therefore  $XYB$ .

THEOREM 3b. *Proof of 3 from A, B, C, 2<sup>3</sup>.*

To prove:  $XAB \cdot AYB \cdot \supset \cdot XYB$ . By B,  $YBX \sim YXB \sim XYB$ .

*Case 1.* Suppose  $YBX$ . Then by 2,  $YBX \cdot BAX \cdot \supset \cdot YBA$ , contrary to  $AYB$ , by C.

*Case 2.* If  $YXB$ , then by 2,  $YXB \cdot XAB \cdot \supset \cdot YXA$ ; whence, by 2,  $BYA \cdot YXA \cdot \supset \cdot BYX$ . Therefore  $XYB$ .

THEOREM 3c. *Proof of 3 from A, C, 2<sup>2</sup>, 6.*

To prove:  $XAB \cdot AYB \cdot \supset \cdot XYB$ . By 2,  $XAB \cdot AYB \cdot \supset \cdot XAY$ .

Hence by 6,  $BAX \cdot YAX \cdot \supset \cdot BYX \sim YBX$ . But if  $YBX$ , then by 2,  $YBX \cdot BAX \cdot \supset \cdot YBA$ , contrary to  $AYB$ , by C. Therefore  $XYB$ .

THEOREM 3d. *Proof of 3 from A, 1, 2.*

To prove:  $XAB \cdot AYB \cdot \supset \cdot XYB$ . By 2,  $XAB \cdot AYB \cdot \supset \cdot XAY$ . Hence by 1,  $BYA \cdot YAX \cdot \supset \cdot BYX$ . Therefore  $XYB$ .

THEOREM 3e. *Proof of 3 from A, C, 2, 8.*

To prove:  $XAB \cdot AYB \cdot \supset \cdot XYB$ . By 2,  $XAB \cdot AYB \cdot \supset \cdot XAY$ . Hence by 8,  $YAX \cdot BAX \cdot \supset \cdot YBA \sim BYX$ . But  $YBA$  is contrary to  $AYB$ , by C. Therefore  $XYB$ .

#### PROOFS OF POSTULATE 4

THEOREM 4a. *Proof of 4 from A, B, C, 1.*

To prove:  $AXB \cdot AYB \cdot \supset \cdot AXY \sim AYX$ . By B,  $AXY \sim AYX \sim XAY$ . Suppose  $XAY$ . Then by 1,  $XAY \cdot AYB \cdot \supset \cdot XAB$ , contrary to  $AXB$ , by C. Therefore  $AXY \sim AYX$ .

THEOREM 4b. *Proof of 4 from A, B, 1, 2.*

To prove:  $AXB \cdot AYB \cdot \supset \cdot AXY \sim AYX$ . By B,  $AXY \sim AYX \sim XAY$ . But if  $XAY$ , then by 1,  $BYA \cdot YAX \cdot \supset \cdot BYX$ , whence, by 2,  $AXB \cdot XYB \cdot \supset \cdot AXY$ . Therefore  $AXY \sim AYX$ .

THEOREM 4c. *Proof of 4 from A, B, 1<sup>2</sup>, 7.*

To prove:  $AXB \cdot AYB \cdot \supset \cdot AXY \sim AYX$ . By B,  $AXY \sim AYX \sim YAX$ . But if  $XAY$ , then by 1,  $XAY \cdot AYB \cdot \supset \cdot XAB$ ; and by 1,  $YAX \cdot AXB \cdot \supset \cdot YAB$ ; whence by 7,  $XAB \cdot YAB \cdot \supset \cdot XYA \sim YXA$ . Therefore  $AXY \sim AYX$ .

THEOREM 4d. *Proof of 4 from A, C, 5<sup>2</sup>.*

To prove:  $AXB \cdot AYB \cdot \supset \cdot AXY \sim AYX$ . By 5,  $AXB \cdot AYB \cdot \supset \cdot AXY \sim YXB$ ; and by 5,  $AYB \cdot AXB \cdot \supset \cdot AYX \sim XYB$ . Suppose  $AXY$  and  $AYX$  are both false. Then  $YXB$  and  $XYB$ , contrary to C. Therefore  $AXY \sim AYX$ .

THEOREM 4e. *Proof of 4 from A, 3<sup>2</sup>, 5<sup>2</sup>, 7<sup>2</sup>.*

To prove:  $AXB \cdot AYB \cdot \supset \cdot AXY \sim AYX$ . By 5,  $AXB \cdot AYB \cdot \supset \cdot AXY \sim YXB$ , and by 5,  $AYB \cdot AXB \cdot \supset \cdot AYX \sim XYB$ . Suppose  $AXY$  and  $AYX$  are both false. Then  $YXB$  and  $XYB$ , whence by 7,  $XYB \cdot AYB \cdot \supset \cdot AXY \sim XAY$ . But if  $XAY$ , then by 3,  $BXY \cdot XAY \cdot \supset \cdot BAY$ , and by 3,  $BYX \cdot YAX \cdot \supset \cdot BAX$ ; whence by 7,  $XAB \cdot YAB \cdot \supset \cdot XYA \sim YXA$ . Therefore  $AXY \sim AYX$ .

THEOREM 4f. *Proof of 4 from A, 5<sup>2</sup>, 7, 8<sup>2</sup>.*

To prove:  $AXB \cdot AYB \cdot \supset \cdot AXY \sim AYX$ .

By 5,  $AXB \cdot AYB \cdot \supset \cdot AXY \sim YXB$ ,  
and by 5,  $AYB \cdot AXB \cdot \supset \cdot AYX \sim XYB$ .

Suppose  $AXY$  and  $AYX$  are both false.

Then  $YXB$  and  $XYB$ , whence by 8,  $AXB \cdot YXB \cdot \supset \cdot AYX \sim YAB$ ,  
and by 8,  $AYB \cdot XYB \cdot \supset \cdot AXY \sim XAB$ .

But if  $YAB$  and  $XAB$ , then by 7,  $XAB \cdot YAB \cdot \supset \cdot XYA \sim YXA$ .  
Therefore  $AXY \sim AYX$ .

THEOREM 4g. *Proof of 4 from 2, 5.*

To prove:  $AXB \cdot AYB \cdot \supset \cdot AXY \sim AYX$ .

By 5,  $AXB \cdot AYB \cdot \supset \cdot AXY \sim YXB$ .

But if  $YXB$ , then by 2,  $AYB \cdot YXB \cdot \supset \cdot AXY$ . Therefore  $AXY \sim AYX$ .

THEOREM 4h. *Proof of 4 from A, 1<sup>2</sup>, 5, 7<sup>2</sup>.*

To prove:  $AXB \cdot AYB \cdot \supset \cdot AXY \sim AYX$ .

By 5,  $AXB \cdot AYB \cdot \supset \cdot AXY \sim YXB$ .

Suppose  $AXY$  false.

Then  $YXB$ , whence, by 7,  $YXB \cdot AXB \cdot \supset \cdot YAX \sim AYX$ .

But if  $YAX$ , then by 1,  $YAX \cdot AXB \cdot \supset \cdot YAB$ ,

and by 1,  $XAY \cdot AYB \cdot \supset \cdot XAB$ ,

whence by 7,  $XAB \cdot YAB \cdot \supset \cdot XYA \sim YXA$ . Therefore  $AXY \sim AYX$ .

The following two theorems, 4i and 4j, are the only ones in which Postulate C is used without Postulate A. (It will be noted that there are no cases in which Postulate B is used without Postulate A.)

THEOREM 4i. *Proof of 4 from C<sup>2</sup>, 5<sup>2</sup>, 7<sup>3</sup>, 8<sup>2</sup>.*

To prove:  $AXB \cdot AYB \cdot \supset \cdot AXY \sim AYX$ .

By 5,  $AXB \cdot AYB \cdot \supset \cdot AXY \sim YXB$ ,

and by 5,  $AYB \cdot AXB \cdot \supset \cdot AYX \sim XYB$ .

Suppose  $AXY$  and  $AYX$  are both false. Then  $YXB$  and  $XYB$ ,

whence by 7,  $YXB \cdot AXB \cdot \supset \cdot YAX \sim AYX$ ,

and by 7,  $XYB \cdot AYB \cdot \supset \cdot XAY \sim AXY$ ; whence  $YAX$  and  $XAY$ .

Again, by 8,  $AXB \cdot YXB \cdot \supset \cdot AYX \sim YAB$ ,

and by 8,  $AYB \cdot XYB \cdot \supset \cdot AXY \sim XAB$ , whence  $YAB$  and  $XAB$ ;

whence, by 7,  $YAB \cdot XAB \cdot \supset \cdot YXA \sim XYA$ , contrary to  $YAX$  and  $XAY$ , respectively, by C. Therefore  $AXY \sim AYX$ .

THEOREM 4j. *Proof of 4 from C<sup>2</sup>, 1<sup>2</sup>, 5<sup>2</sup>, 7<sup>3</sup>.*

To prove:  $AXB \cdot AYB \cdot \supset \cdot AXY \sim AYX$ .

By 5,  $AXB \cdot AYB \cdot \supset \cdot AXY \sim YXB$ ,

and by 5,  $AYB \cdot AXB \cdot \supset \cdot AYX \sim XYB$ .

Suppose  $AXY$  and  $AYX$  are both false. Then  $YXB$  and  $XYB$ ,



whence by 7,  $AXB \cdot YXB \cdot \supset AYX \sim YAX$ ,

and by 7,  $XYB \cdot AYB \cdot \supset XAY \sim AXY$ .

But if  $YAX$  and  $XAY$ , then by 1,  $YAX \cdot AXB \cdot \supset YAB$ ,

and by 1,  $XAY \cdot AYB \cdot \supset XAB$ ,

whence by 7,  $YAB \cdot XAB \cdot \supset YXA \sim XYA$ , contrary to  $YAX$  and  $XAY$ , respectively, by C. Therefore  $AXY \sim AYX$ .

#### PROOFS OF POSTULATE 5

THEOREM 5a. *Proof of 5 from A, B, 1, 2.*

To prove:  $AXB \cdot AYB \cdot \supset AXY \sim YXB$ . By B,  $AXY \sim XYA \sim XAY$ . But if  $XYA$ , then by 2,  $BXA \cdot XYA \cdot \supset BXY$ ; and if  $XAY$ , then by 1,  $BXA \cdot XAY \cdot \supset BXY$ . Therefore  $AXY \sim YXB$ .

THEOREM 5b. *Proof of 5 from A, B, 1<sup>2</sup>, 7.*

To prove:  $AXB \cdot AYB \cdot \supset AXY \sim YXB$ . By B,  $AXY \sim XAY \sim XYA$ .

Case 1. If  $XAY$ , then by 1,  $BXA \cdot XAY \cdot \supset BXY$ .

Case 2. If  $XYA$ , then by 7,  $XYA \cdot BYA \cdot \supset XBY \sim BXY$ . But if  $XBY$ , then by 1,  $AXB \cdot XBY \cdot \supset AXY$ . Therefore  $AXY \sim YXB$ .

THEOREM 5c. *Proof of 5 from A, B, C, 1, 8.*

To prove:  $AXB \cdot AYB \cdot \supset AXY \sim YXB$ . By B,  $AXY \sim XAY \sim XYA$ .

Case 1. Suppose  $XAY$ ; then by 1,  $BXA \cdot XAY \cdot \supset BXY$ .

Case 2. Suppose  $XYA$ ; then by 8,  $BYA \cdot XYA \cdot \supset BXY \sim XBA$ , where  $XBA$  is contrary to  $AXB$ , by C. Therefore  $AXY \sim YXB$ .

THEOREM 5d. *Proof of 5 by A, B<sup>2</sup>, C<sup>3</sup>, 1<sup>2</sup>, 6.*

To prove:  $AXB \cdot AYB \cdot \supset AXY \sim YXB$ . By B,  $AXY \sim XAY \sim XYA$ , and by B,  $YXB \sim BYX \sim YBX$ .

Case 1. If  $XAY$ , then by 1,  $BXA \cdot XAY \cdot \supset BXY$ .

Case 2. If  $XYA$  and  $BYX$ , then by 6,  $AYX \cdot BYX \cdot \supset ABX \sim BAX$ , contrary to  $AXB$ , by C.

Case 3. If  $XYA$  and  $YBX$ , then by 1,  $YBX \cdot BXA \cdot \supset YBA$ , contrary to  $AYB$ , by C. Therefore  $AXY \sim YXB$ .

THEOREM 5e. *Proof of 5 from A, 2, 4.*

To prove:  $AXB \cdot AYB \cdot \supset AXY \sim YXB$ .

By 4,  $AXB \cdot AYB \cdot \supset AXY \sim AYX$ .

But if  $AYX$ , then by 2,  $BXA \cdot XYA \cdot \supset BXY$ . Therefore  $AXY \sim YXB$ .

THEOREM 5f. *Proof of 5 from A, C, 4<sup>2</sup>, 7.*

To prove:  $AXB \cdot AYB \cdot \supset AXY \sim YXB$ .

By 4,  $AXB \cdot AYB \cdot \supset AXY \sim AYX$ ;

and by 4,  $BXA \cdot BYA \cdot \supset BXY \sim BYX$ .

Suppose  $AYX$  and  $BYX$ . Then by 7,  $XYA \cdot BYA \cdot \supset XBY \sim BXY$ , where  $XBY$  is contrary to  $BYX$ , by C. Therefore  $AXY \sim YXB$ .

THEOREM 5g. *Proof of 5 from A, C<sup>2</sup>, 4<sup>2</sup>, 6.*

To prove:  $AXB \cdot AYB \cdot \supset \cdot AXY \sim YXB$ .

By 4,  $AXB \cdot AYB \cdot \supset \cdot AXY \sim AYX$ ;

and by 4,  $BXA \cdot BYA \cdot \supset \cdot BXY \sim BYX$ .

Suppose  $AXY$  and  $YXB$  are both false.

Then  $AYX$  and  $BYX$ , whence by 6,  $AYX \cdot BYX \cdot \supset \cdot ABX \sim BAX$ ,  
contrary to  $AXB$ , by C. Therefore  $AXY \sim YXB$ .

THEOREM 5h. *Proof of 5 from A, 1, 4, 7.*

To prove:  $AXB \cdot AYB \cdot \supset \cdot AXY \sim YXB$ .

By 4,  $AXB \cdot AYB \cdot \supset \cdot AXY \sim AYX$ .

If  $AYX$ , then by 7,  $XYA \cdot BYA \cdot \supset \cdot XBY \sim BXY$ . But if  $XBY$ , then by 1,  $AXB \cdot XBY \cdot \supset \cdot AXY$ . Therefore  $AXY \sim YXB$ .

THEOREM 5i. *Proof of 5 from A, C, 4, 8.*

To prove:  $AXB \cdot AYB \cdot \supset \cdot AXY \sim YXB$ .

By 4,  $AXB \cdot AYB \cdot \supset \cdot AXY \sim AYX$ .

But if  $AYX$ , then by 8,  $BYA \cdot XYA \cdot \supset \cdot BXY \sim XBA$ , where  $XBA$  is  
contrary to  $AXB$ , by C. Therefore  $AXY \sim YXB$ .

THEOREM 5j. *Proof of 5 from A, 3<sup>2</sup>, 4<sup>2</sup>, 7.*

To prove:  $AXB \cdot AYB \cdot \supset \cdot AXY \sim YXB$ .

By 4,  $AXB \cdot AYB \cdot \supset \cdot AXY \sim AYX$ ,

and by 4,  $BXA \cdot BYA \cdot \supset \cdot BXY \sim BYX$ .

Suppose  $AXY$  and  $YXB$  are both false.

Then  $AYX$  and  $BYX$ , whence by 7,  $AYX \cdot BYX \cdot \supset \cdot ABY \sim BAY$ .

But if  $ABY$ , then by 3,  $YBA \cdot BXA \cdot \supset \cdot YXA$ ; and if  $BAY$ , then by 3,  
 $YAB \cdot AXB \cdot \supset \cdot YXB$ . Therefore  $AXY \sim YXB$ .

#### PROOFS OF POSTULATE 6

THEOREM 6a. *Proof of 6 from A, B, C, 2.*

To prove:  $XAB \cdot YAB \cdot \supset \cdot XYB \sim YXB$ . By B,  $XYB \sim YBX \sim BXY$ .  
Suppose  $XBY$ . Then by 2,  $XBY \cdot BAY \cdot \supset \cdot XBA$ , contrary to  $XAB$ ,  
by C. Therefore  $XYB \sim YXB$ .

THEOREM 6b. *Proof of 6 from A, B, 2<sup>2</sup>, 7.*

To prove:  $XAB \cdot YAB \cdot \supset \cdot XYB \sim YXB$ . By B,  $XYB \sim YBX \sim BXY$ .  
But if  $XBY$ , then by 2,  $XBY \cdot BAY \cdot \supset \cdot XBA$ ;  
and by 2,  $YBX \cdot BAX \cdot \supset \cdot YBA$ ;  
whence by 7,  $XBA \cdot YBA \cdot \supset \cdot XYB \sim YXB$ . Therefore  $XYB \sim YXB$ .

THEOREM 6c. *Proof of 6 from 1<sup>2</sup>, 7.*

To prove:  $XAB \cdot YAB \cdot \supset \cdot XYB \sim YXB$ .

By 7,  $XAB \cdot YAB \cdot \supset \cdot XYA \sim YXA$ .

Case 1. If  $XYA$ , then by 1,  $XYA \cdot YAB \cdot \supset \cdot XYB$ .

*Case 2.* If  $YXA$ , then by 1,  $YXA \cdot XAB \cdot \supset \cdot YXB$ .  
Therefore  $XYB \sim YXB$ .

THEOREM 6d. *Proof of 6 from A, 3<sup>2</sup>, 7.*

To prove:  $XAB \cdot YAB \cdot \supset \cdot XYB \sim YXB$ .

By 7,  $XAB \cdot YAB \cdot \supset \cdot XYA \sim YXA$ .

*Case 1.* If  $XYA$ , then by 3,  $BAX \cdot AYX \cdot \supset \cdot BYX$ .

*Case 2.* If  $YXA$ , then by 3,  $BAY \cdot AXY \cdot \supset \cdot BXY$ .

Therefore  $XYB \sim YXB$ .

THEOREM 6e. *Proof of 6 from 1, 8.*

To prove:  $XAB \cdot YAB \cdot \supset \cdot XYB \sim YXB$ .

By 8,  $XAB \cdot YAB \cdot \supset \cdot XYA \sim YXB$ .

But if  $XYA$ , then by 1,  $XYA \cdot YAB \cdot \supset \cdot XYB$ .

Therefore  $XYB \sim YXB$ .

THEOREM 6f. *Proof of 6 from A, 3, 8.*

To prove:  $XAB \cdot YAB \cdot \supset \cdot XYB \sim YXB$ .

By 8,  $XAB \cdot YAB \cdot \supset \cdot XYA \sim YXB$ .

But if  $XYA$ , then by 3,  $BAX \cdot AYX \cdot \supset \cdot BYX$ .

Therefore  $XYB \sim YXB$ .

THEOREM 6g. *Proof of 6 from A, B, 2<sup>2</sup>, 8.*

To prove:  $XAB \cdot YAB \cdot \supset \cdot XYB \sim YXB$ . By B,  $XYB \sim YXB \sim YBX$ , and by 8,  $XAB \cdot YAB \cdot \supset \cdot XYA \sim YXB$ . But if  $YBX$  and  $XYA$ , then by 2,  $YBX \cdot BAX \cdot \supset \cdot YBA$ , whence by 2,  $XYA \cdot YBA \cdot \supset \cdot XYB$ . Therefore  $XYB \sim YXB$ .

THEOREM 6h. *Proof of 6 from A, B, C<sup>2</sup>, 3, 5.*

To prove:  $XAB \cdot YAB \cdot \supset \cdot XYB \sim YXB$ . By B,  $XYB \sim YBX \sim BXY$ . Suppose  $YBX$ . Then by 3,  $YBX \cdot BAX \cdot \supset \cdot YAX$ , whence by 5,  $YBX \cdot YAX \cdot \supset \cdot YBA \sim ABX$ , contrary to  $YAB$  and  $XAB$ , respectively, by C. Therefore  $XYB \sim YXB$ .

THEOREM 6i. *Proof of 6 from A, C, 8<sup>2</sup>.*

To prove:  $XAB \cdot YAB \cdot \supset \cdot XYB \sim YXB$ .

By 8,  $XAB \cdot YAB \cdot \supset \cdot XYA \sim YXB$ ;

and by 8,  $YAB \cdot XAB \cdot \supset \cdot YXA \sim XYB$ .

Suppose  $YXB$  and  $XYB$  are both false. Then  $XYA$  and  $YXA$ , contrary to C. Therefore  $XYB \sim YXB$ .

THEOREM 6j. *Proof of 6 from A, B<sup>2</sup>, C<sup>2</sup>, 1<sup>2</sup>, 5.*

To prove:  $XAB \cdot YAB \cdot \supset \cdot XYB \sim YXB$ . By C,  $XBA$  and  $ABY$  are false, since  $XAB$  and  $YAB$  are true. By B,  $XAY \sim XYA \sim YXA$ ; and by B,  $XBY \sim XYB \sim YXB$ .

*Case 1.* Suppose  $XAY$  and  $XBY$ .

Then by 5,  $XBY \cdot XAY \cdot \supset \cdot XBA \sim ABY$ , which are both false.

*Case 2.* If  $XYA$ , then by 1,  $XYA \cdot YAB \supset XYB$ .

*Case 3.* If  $YXA$ , then by 1,  $YXA \cdot XAB \supset YXB$ .

Therefore  $XYB \sim YXB$ .

#### PROOFS OF POSTULATE 7

**THEOREM 7a.** *Proof of 7 from A, B, C, 2<sup>3</sup>.*

To prove:  $XAB \cdot YAB \supset XYA \sim YXA$ . By B,  $XYB \sim YXB \sim XBY$ .

*Case 1.* If  $XYB$ , then by 2,  $XYB \cdot YAB \supset XYA$ .

*Case 2.* If  $YXB$ , then by 2,  $YXB \cdot XAB \supset YXA$ .

*Case 3.* If  $XBY$ , then by 2,  $XBY \cdot BAY \supset XBA$ , contrary to  $XAB$ , by C. Therefore  $XYA \sim YXA$ .

**THEOREM 7b.** *Proof of 7 from A, B, C<sup>3</sup>, 6<sup>3</sup>.*

To prove:  $XAB \cdot YAB \supset XYA \sim YXA$ .

By 6,  $XAB \cdot YAB \supset XYB \sim YXB$ . Hence by C,  $YBX$  is false.

By B,  $XAY \sim XYA \sim YXA$ . Suppose  $XAY$ .

Then by 6,  $YAX \cdot BAX \supset YBX \sim BYX$ ;

and by 6,  $XAY \cdot BAY \supset XBY \sim BXY$ . But  $YBX$  is false; hence both  $BYX$  and  $BXY$  must be true, which is impossible by C.

Therefore  $XYA \sim YXA$ .

**THEOREM 7c.** *Proof of 7 from A, C<sup>3</sup>, 4<sup>2</sup>, 6<sup>3</sup>.*

To prove:  $XAB \cdot YAB \supset XYA \sim YXA$ .

By 6,  $XAB \cdot YAB \supset XYB \sim YXB$ .

*Case 1.* Suppose  $XYB$  true. Then by 4,  $XAB \cdot XYB \supset XAY \sim XYA$ . But if  $XAY$ , then by 6,  $XAY \cdot BAY \supset XBY \sim BXY$ , contrary to  $XYB$  by C. Hence in Case 1,  $XYA$ .

*Case 2.* Suppose  $XYB$  false; then  $YXB$ .

Then by 4,  $YAB \cdot YXB \supset YAX \sim YXA$ .

But if  $YAX$ , then by 6,  $YAX \cdot BAX \supset YBX \sim BYX$ , where  $BYX$  is false by hypothesis, and  $YBX$  is contrary to  $YXB$  by C. Hence in Case 2,  $YXA$ . Therefore  $XYA \sim YXA$ .

**THEOREM 7d.** *Proof of 7 from 2<sup>2</sup>, 6.*

To prove:  $XAB \cdot YAB \supset XYA \sim YXA$ .

By 6,  $XAB \cdot YAB \supset XYB \sim YXB$ .

*Case 1.* If  $XYB$ , then by 2,  $XYB \cdot YAB \supset XYA$ .

*Case 2.* If  $YXB$ , then by 2,  $YXB \cdot XAB \supset YXA$ .

Hence in either case,  $XYA \sim YXA$ .

**THEOREM 7e.** *Proof of 7 from 2, 8.*

To prove:  $XAB \cdot YAB \supset XYA \sim YXA$ .

By 8,  $XAB \cdot YAB \supset XYA \sim YXB$ .

If  $YXB$ , then by 2,  $YXB \cdot XAB \supset YXA$ .

Therefore  $XYA \sim YXA$ .

THEOREM 7f. *Proof of 7 from A, C, 8<sup>2</sup>.*

To prove:  $XAB \cdot YAB \cdot \supset \cdot XYA \sim YXA$ .

By 8,  $XAB \cdot YAB \cdot \supset \cdot XYA \sim YXB$ ;

and by 8,  $YAB \cdot XAB \cdot \supset \cdot YXA \sim XYB$ .

Suppose  $XYA$  and  $YXA$  are both false. Then  $YXB$  and  $XYB$ , contrary to C. Therefore  $XYA \sim YXA$ .

THEOREM 7g. *Proof of 7 from A, B<sup>2</sup>, C<sup>4</sup>, 5<sup>3</sup>.*

To prove:  $XAB \cdot YAB \cdot \supset \cdot XYA \sim YXA$ . By B,  $XYB \sim YXB \sim XBY$ , and by B,  $XAY \sim XYA \sim YXA$ .

Case 1. If  $XYB$ , then by 5,  $XYB \cdot XAB \cdot \supset \cdot XYA \sim AYB$ , where  $AYB$  is contrary to  $YAB$ , by C. Hence in Case 1,  $XYA$ .

Case 2. If  $YXB$ , then by 5,  $YXB \cdot YAB \cdot \supset \cdot YXA \sim AXB$ , where  $AXB$  is contrary to  $XAB$ , by C. Hence in Case 2,  $YXA$ .

Case 3. If  $XBY$  and  $XAY$ , then by 5,  $XBY \cdot XAY \cdot \supset \cdot XBA \sim ABY$ , contrary to  $XAB$  and  $YAB$ , by C. Therefore  $XYA \sim YXA$ .

THEOREM 7h. *Proof of 7 from A, C<sup>2</sup>, 5<sup>2</sup>, 6.*

To prove:  $XAB \cdot YAB \cdot \supset \cdot XYA \sim YXA$ .

By 6,  $XAB \cdot YAB \cdot \supset \cdot XYB \sim YXB$ .

Case 1. If  $XYB$ , then by 5,  $XYB \cdot XAB \cdot \supset \cdot XYA \sim AYB$ , where  $AYB$  is contrary to  $YAB$ , by C. Hence in Case 1,  $XYA$ .

Case 2. If  $YXB$ , then by 5,  $YXB \cdot YAB \cdot \supset \cdot YXA \sim AXB$ , where  $AXB$  is contrary to  $XAB$ , by C. Hence in Case 2,  $YXA$ . Therefore  $XYA \sim YXA$ .

THEOREM 7i. *Proof of 7 from A, 4, 5<sup>2</sup>, 8<sup>2</sup>.*

To prove:  $XAB \cdot YAB \cdot \supset \cdot XYA \sim YXA$ .

By 8,  $XAB \cdot YAB \cdot \supset \cdot XYA \sim YXB$ ;

and by 8,  $YAB \cdot XAB \cdot \supset \cdot YXA \sim XYB$ .

Suppose  $XYA$  and  $YXA$  are both false; then  $YXB$  and  $XYB$ , whence by 5,  $YXB \cdot YAB \cdot \supset \cdot YXA \sim AXB$ , and by 5,  $XYB \cdot XAB \cdot \supset \cdot XYA \sim AYB$ .

But if  $AXB$  and  $AYB$ , then by 4,  $AXB \cdot AYB \cdot \supset \cdot AXY \sim AYX$ .

Therefore  $XYA \sim YXA$ .

THEOREM 7j. *Proof of 7 from A, 1<sup>2</sup>, 4<sup>3</sup>, 5<sup>2</sup>, 6<sup>3</sup>.*

To prove:  $XAB \cdot YAB \cdot \supset \cdot XYA \sim YXA$ .

By 6,  $XAB \cdot YAB \cdot \supset \cdot XYB \sim YXB$ .

If  $XYB$  is true, then by 4,  $XYB \cdot XAB \cdot \supset \cdot XYA \sim XAY$ ;

and by 5,  $XYB \cdot XAB \cdot \supset \cdot XYA \sim AYB$ .

If  $YXB$  is true, then by 4,  $YXB \cdot YAB \cdot \supset \cdot YXA \sim YAX$ ,

and by 5,  $YXB \cdot YAB \cdot \supset \cdot YXA \sim AXB$ . Suppose that  $XYA$  and  $YXA$  are both false. Then there are three cases to consider.

Case 1. Suppose  $XYB$  true and  $YXB$  false. Then  $XAY$  and  $AYB$ .

Then by 6,  $XAY \cdot BAY \cdot \supset \cdot XBY \sim BXY$ ,  
whence by 1,  $AYB \cdot YBX \cdot \supset \cdot AXY$ .

*Case 2.* Suppose  $XYB$  false and  $YXB$  true. Then  $YAX$  and  $AXB$ .  
Then by 6,  $YAX \cdot BAX \cdot \supset \cdot YBX \sim BYX$ ,  
whence by 1,  $AXB \cdot XBY \cdot \supset \cdot AXY$ .

*Case 3.* Suppose  $XYB$  true and  $YXB$  true. Then  $AYB$  and  $AXB$ .  
Then by 4,  $AXB \cdot AYB \cdot \supset \cdot AXY \sim AYX$ . Therefore  $XYA \sim YXA$ .

#### PROOFS OF POSTULATE 8

**THEOREM 8a.** *Proof of 8 from A, B, C, 2<sup>2</sup>.*

To prove:  $XAB \cdot YAB \cdot \supset \cdot XYA \sim YXB$ . By B,  $XBY \sim XYB \sim YXB$ .

*Case 1.* If  $XBY$ , then by 2,  $XBY \cdot BAY \cdot \supset \cdot XBA$ , contrary to  $XAB$ ,  
by C.

*Case 2.* If  $XYB$ , then by 2,  $XYB \cdot YAB \cdot \supset \cdot XYA$ .

Therefore  $XYA \sim YXB$ .

**THEOREM 8b.** *Proof of 8 from A, B<sup>2</sup>, C<sup>3</sup>, 1, 5<sup>2</sup>.*

To prove:  $XAB \cdot YAB \cdot \supset \cdot XYA \sim YXB$ . By C,  $XBA$  and  $ABY$  and  
 $AYB$  are false, since  $XAB$  and  $YAB$  are true. By B,  $XAY \sim YXA \sim XYA$ ;  
and by B,  $XBY \sim XYB \sim YXB$ .

*Case 1.* Suppose  $XAY$  and  $XBY$ .

Then by 5,  $XBY \cdot XAY \cdot \supset \cdot XBA \sim ABY$ , which are both false.

*Case 2.* If  $XYB$ , then by 5,  $XYB \cdot XAB \cdot \supset \cdot XYA \sim AYB$ , where  $AYB$   
is false.

*Case 3.* If  $YXA$ , then by 1,  $YXA \cdot XAB \cdot \supset \cdot YXB$ .

Therefore  $XYA \sim YXB$ .

**THEOREM 8c.** *Proof of 8 from A, B<sup>2</sup>, C<sup>3</sup>, 3, 5<sup>2</sup>.*

To prove:  $XAB \cdot YAB \cdot \supset \cdot XYA \sim YXB$ . By C,  $XBA$  and  $ABY$  and  
 $AYB$  are false, since  $XAB$  and  $YAB$  are true. By B,  $XAY \sim YXA \sim XYA$ ;  
and by B,  $XBY \sim XYB \sim YXB$ .

*Case 1.* Suppose  $XAY$  and  $XBY$ .

Then by 5,  $XBY \cdot XAY \cdot \supset \cdot XBA \sim ABY$ , which are both false.

*Case 2.* If  $XYB$ , then by 5,  $XYB \cdot XAB \cdot \supset \cdot XYA \sim AYB$ , where  
 $AYB$  is false.

*Case 3.* If  $YXA$ , then by 3,  $BAY \cdot AXY \cdot \supset \cdot BXY$ .

Therefore  $XYA \sim YXB$ .

**THEOREM 8d.** *Proof of 8 from A, B, C, 3, 6<sup>2</sup>.*

To prove:  $XAB \cdot YAB \cdot \supset \cdot XYA \sim YXB$ .

By 6,  $XAB \cdot YAB \cdot \supset \cdot XYB \sim YXB$ ; and by B,  $AXY \sim XAY \sim XYA$ .

*Case 1.* If  $AXY$ , then by 3,  $BAY \cdot AXY \cdot \supset \cdot BXY$ . Hence in Case 1,  
 $YXB$ .

*Case 2.* If  $XAY$  and  $XYB$ , then by 6,  $XAY \cdot BAY \cdot \supset \cdot XBY \sim BXY$ , where  $XBY$  is contrary to  $XYB$ , by C. Hence in Case 2,  $YXB$ .

Therefore  $XYA \sim YXB$ .

THEOREM 8e. *Proof of 8 from 1, 7.*

To prove:  $XAB \cdot YAB \cdot \supset \cdot XYA \sim YXB$ .

By 7,  $XAB \cdot YAB \cdot \supset \cdot XYA \sim YXA$ .

But if  $YXA$ , then by 1,  $YXA \cdot XAB \cdot \supset \cdot YXB$ .

Therefore  $XYA \sim YXB$ .

THEOREM 8f. *Proof of 8 from A, B, C, 1, 6<sup>2</sup>.*

To prove:  $XAB \cdot YAB \cdot \supset \cdot XYA \sim YXB$ .

By 6,  $XAB \cdot YAB \cdot \supset \cdot XYB \sim YXB$ ; and by B,  $YXA \sim XAY \sim XYA$ .

*Case 1.* If  $YXA$ , then by 1,  $YXA \cdot XAB \cdot \supset \cdot YXB$ . Hence in Case 1,  $YXB$ .

*Case 2.* If  $XAY$  and  $XYB$ , then by 6,  $XAY \cdot BAY \cdot \supset \cdot XBY \sim BXY$ , where  $XBY$  is contrary to  $XYB$ , by C. Hence in Case 2,  $YXB$ . Therefore  $XYA \sim YXB$ .

THEOREM 8g. *Proof of 8 from A, 3, 7.*

To prove:  $XAB \cdot YAB \cdot \supset \cdot XYA \sim YXB$ .

By 7,  $XAB \cdot YAB \cdot \supset \cdot XYA \sim YXA$ .

But if  $YXA$ , then by 3,  $BAY \cdot AXY \cdot \supset \cdot BXY$ .

Therefore  $XYA \sim YXB$ .

THEOREM 8h. *Proof of 8 from 2, 6.*

To prove:  $XAB \cdot YAB \cdot \supset \cdot XYA \sim YXB$ .

By 6,  $XAB \cdot YAB \cdot \supset \cdot XYB \sim YXB$ .

But if  $XYB$ , then by 2,  $XYB \cdot YAB \cdot \supset \cdot XYA$ .

Therefore  $XYA \sim YXB$ .

THEOREM 8i. *Proof of 8 from A, C, 5, 6.*

To prove:  $XAB \cdot YAB \cdot \supset \cdot XYA \sim YXB$ .

By 6,  $XAB \cdot YAB \cdot \supset \cdot XYB \sim YXB$ .

But if  $XYB$ , then by 5,  $XYB \cdot XAB \cdot \supset \cdot XYA \sim AYB$ , where  $AYB$  is contrary to  $YAB$ , by C. Therefore  $XYA \sim YXB$ .

THEOREM 8j. *Proof of 8 from A, C, 4, 6<sup>2</sup>.*

To prove:  $XAB \cdot YAB \cdot \supset \cdot XYA \sim YXB$ .

By 6,  $XAB \cdot YAB \cdot \supset \cdot XYB \sim YXB$ . Suppose  $XYB$ .

Then by 4,  $XAB \cdot XYB \cdot \supset \cdot XAY \sim XYA$ . But if  $XAY$ , then by 6,  $XAY \cdot BAY \cdot \supset \cdot XBY \sim BXY$ , where  $XBY$  is contrary to  $XYB$ , by C.

Therefore  $XYA \sim YXB$ .

THEOREM 8k. *Proof of 8 from A, B, 2<sup>3</sup>, 7.*

To prove:  $XAB \cdot YAB \cdot \supset \cdot XYA \sim YXB$ . By B,  $XBY \sim XYB \sim YXB$ .

*Case 1.* If  $XBY$ , then by 2,  $YBX \cdot BAX \cdot \supset \cdot YBA$ ,

and by 2,  $XYB \cdot BAY \cdot \supset \cdot XBA$ ,

whence by 7,  $XBA \cdot YBA \cdot \supset \cdot XYB \sim YXB$ .

*Case 2.* If  $XYB$ , then by 2,  $XYB \cdot YAB \cdot \supset \cdot XYA$ .

Therefore  $XYA \sim YXB$ .

**THEOREM 8l.** *Proof of 8 from A, B, 1<sup>2</sup>, 5, 6<sup>2</sup>.*

To prove:  $XAB \cdot YAB \cdot \supset \cdot XYA \sim YXB$ . By B,  $YXA \sim XAY \sim XYA$ , and by 6,  $XAB \cdot YAB \cdot \supset \cdot XYB \sim YXB$ .

*Case 1.* If  $YXA$ , then by 1,  $YXA \cdot XAB \cdot \supset \cdot YXB$ .

*Case 2.* If  $XAY$  and  $XYB$ , then by 5,  $XYB \cdot XAB \cdot \supset \cdot XYA \sim AYB$ , and by 6,  $XAY \cdot BAY \cdot \supset \cdot XBY \sim BXY$ . But if  $AYB$  and  $XBY$ , then by 1,  $AYB \cdot YBX \cdot \supset \cdot AYX$ . Therefore  $XYA \sim YXB$ .

**THEOREM 8m.** *Proof of 8 from A, 1, 4, 5, 6<sup>2</sup>.*

To prove:  $XAB \cdot YAB \cdot \supset \cdot XYA \sim YXB$ . Suppose  $XYA$  and  $YXB$  are both false. Now by 6,  $XAB \cdot YAB \cdot \supset \cdot XYB \sim YXB$ . As  $YXB$  is false,  $XYB$ . By 4,  $XYB \cdot XAB \cdot \supset \cdot XYA \sim XAY$ . As  $XYA$  is false,  $XAY$ . By 5,  $XYB \cdot XAB \cdot \supset \cdot XYA \sim AYB$ . As  $XYA$  is false,  $AYB$ . By 6,  $BAY \cdot XAY \cdot \supset \cdot BXY \sim XBY$ . As  $BXY$  is false, then  $XBY$ . By 1,  $AYB \cdot YBX \cdot \supset \cdot AYX$ , contrary to supposition. Therefore  $XYA \sim YXB$ .

The results of these 71 theorems may be conveniently summarized in the following table. In this table, the numbers in the last column indicate the sets of independent postulates, if any (see § 4), in connection with which each theorem is available.

TABLE I. THEOREMS ON DEDUCIBILITY

Theorem	Postulate	follows from	Set.
1a	1	A B C 2 4	6
1b	1	A B C 3 4	9, 10, 11
1c	1	A B C 2 5	7
1d	1	A B C 3 5	8
2a	2	A B C 1 7	4
2b	2	A B C 1 6	3
2c	2	A B C 3 6	9
2d	2	A C 3 7	10
2e	2	A C 3 4 6	9
2f	2	A B C 1 8	5
2g	2	A B C 1 5	2
2h	2	A C 3 8	11
2i	2	A C 3 5	8
3a	3	A B C 1	1, 2, 3, 4, 5
3b	3	A B C 2	1, 6, 7
3c	3	A C 2 6	
3d	3	A 1 2	1
3e	3	A C 2 8	
4a	4	A B C 1	1, 2, 3, 4, 5
4b	4	A B 1 2	1
4c	4	A B 1 7	4
4d	4	A C 5 7	2, 7, 8
4e	4	A 3 5 7	



TABLE I—Continued

Theorem	Postulate	follows from								Set.
4f	4	A				5	7	8		
4g	4				2	5				
4h	4	A		1		5	7			7
4i	4		C			5	7	8		
4j	4		C	1		5	7			
5a	5	A	B	1	2					1
5b	5	A	B	1			7			4
5c	5	A	B	C	1			8		5
5d	5	A	B	C	1		6			3
5e	5	A			2	4				6
5f	5	A		C		4	7			10
5g	5	A		C		4	6	7		9
5h	5	A			1	4		7	8	
5i	5	A		C		4			8	11
5j	5	A				3	4	7		10
6a	6	A	B	C	2					1, 6, 7
6b	6	A	B		2			7		
6c	6			1				7		4
6d	6	A				3		7		10
6e	6			1					8	5
6f	6	A				3			8	11
6g	6	A	B		2				8	
6h	6	A	B	C		3	5			8
6i	6	A		C					8	5, 11
6j	6	A	B	C	1		5			2
7a	7	A	B	C	2					1, 6, 7
7b	7	A	B	C			6			3, 9
7c	7	A		C		4	6			9
7d	7				2		6			
7e	7				2				8	
7f	7	A		C					8	
7g	7	A	B	C			5			5, 11
7h	7	A		C			5	6		2, 7, 8
7i	7	A				4	5		8	
7j	7	A			1	4	5	6		
8a	8	A	B	C	2					1, 6, 7
8b	8	A	B	C	1		5			2
8c	8	A	B	C		3	5			8
8d	8	A	B	C		3		6		9
8e	8			1					7	4
8f	8	A	B	C	1			6		3
8g	8	A				3			7	10
8h	8				2			6		
8i	8	A		C			5	6		
8j	8	A		C			4	6		9
8k	8	A	B		2				7	
8l	8	A	B	1			5	6		
8m	8	A			1	4	5	6		

## 3. THEOREMS ON NON-DEDUCIBILITY AND EXAMPLES OF PSEUDO-BETWEENNESS

In this section we show that the question proposed at the beginning of § 2 must always be answered in the negative, except in the cases covered by the 71 theorems just established. That is, we show that no one of the twelve postulates of our basic list is deducible from any others of the list, except in the cases covered by our 71 theorems.

In order to prove this statement, we first construct 44 *examples of pseudo-betweenness*, that is, 44 examples of systems  $(K, R)$  which satisfy some but not all of the twelve postulates.

In the first four examples, A–D, the class  $K$  consists of three elements.

EXAMPLE A. Let  $K$  = a class of three numbers, say 1, 2, 3, and let  $XYZ$  be true in the cases 123, 231, and false in all other cases.

Here 123 is true, while 321 is false, so that Postulate A is not satisfied. C is satisfied vacuously, since the conditions mentioned in the hypothesis do not occur. B and D hold. Postulates 1–8 are satisfied vacuously, since the class contains only three elements.

EXAMPLE B. Let  $K$  = a class of three numbers, say 1, 2, 3, and let  $XYZ$  be false for all values of  $X, Y, Z$ .

Here B is clearly not satisfied. A, C, D, and 1–8 are satisfied vacuously.

EXAMPLE C. Let  $K$  = a class of three numbers, say 1, 2, 3, and let  $XYZ$  mean that  $X, Y, Z$  are distinct.

Here A, B, and D are satisfied, while C is not. Postulates 1–8 are satisfied vacuously.

EXAMPLE D. Let  $K$  = a class of any three numbers; and let  $XYZ$  mean that  $Y$  belongs to the interval from  $X$  to  $Z$  inclusive, when  $X, Y$ , and  $Z$  are arranged in order of magnitude.

Here A, B, and C are satisfied, while D is not. Postulates 1–8 are satisfied vacuously.

In the remaining examples, 1–40, the class  $K$  consists of four numbers, 1, 2, 3, 4, and the meaning of  $R[XYZ]$  is defined by simply giving a catalog of the ordered triads of elements for which the relation is true.

In all these examples, Postulate D is satisfied.\* Which of the other postulates is satisfied in each case may be ascertained from Table II.

In examples 1–5, all four of the Postulates A, B, C, D are satisfied.

EXAMPLE 1. 124, 134, 213, 243, 312, 342, 421, 431.

EXAMPLE 2. 123, 142, 234, 241, 314, 321, 413, 432.

EXAMPLE 3. 123, 142, 143, 241, 243, 321, 341, 342.

EXAMPLE 4. 142, 213, 234, 241, 312, 314, 413, 432.

EXAMPLE 5. 123, 143, 214, 321, 324, 341, 412, 423.

In Examples 6–11, Postulate B fails, while A, C, and D hold.

EXAMPLE 6. 123, 234, 321, 432.

EXAMPLE 7. 123, 143, 243, 321, 341, 342.

EXAMPLE 8. 123, 243, 321, 342.

EXAMPLE 9. 123, 124, 243, 321, 342, 421.

\* In verifying these examples with respect to Postulates 5 and 8, the following peculiarity of these two postulates should be borne in mind: as to 5, for example, it is not sufficient to test for  $AXB \cdot AYB$ ; it is necessary also to test for  $AYB \cdot AXB$ ; and similarly as to 8. For illustrations of the importance of this precaution, see, for instance, Examples 33 and 34.

EXAMPLE 10. 123, 143, 321, 341.

EXAMPLE 11. 213, 214, 312, 412.

In Examples 12–24, Postulate C fails, while A, B, and D hold.

EXAMPLE 12. 123, 132, 214, 231, 234, 314, 321, 324, 412, 413, 423, 432.

EXAMPLE 13. 124, 132, 134, 213, 214, 231, 234, 312, 314, 324, 412, 413, 421, 423, 431, 432.

EXAMPLE 14. 124, 213, 234, 312, 314, 324, 413, 421, 423, 432.

EXAMPLE 15. 123, 124, 143, 234, 243, 321, 341, 342, 421, 432.

EXAMPLE 16. 123, 143, 213, 214, 243, 312, 314, 321, 324, 341, 342, 412, 413, 423.

EXAMPLE 17. 123, 143, 214, 243, 321, 324, 341, 342, 412, 423.

EXAMPLE 18. 213, 214, 234, 243, 312, 314, 324, 342, 412, 413, 423, 432.

EXAMPLE 19. 123, 124, 132, 134, 231, 234, 321, 324, 421, 423, 431, 432.

EXAMPLE 20. 123, 124, 134, 143, 234, 243, 321, 324, 341, 342, 421, 423, 431, 432.

EXAMPLE 21. 123, 132, 134, 142, 143, 213, 214, 231, 234, 241, 243, 312, 321, 341, 342, 412, 431, 432.

EXAMPLE 22. 123, 124, 132, 134, 143, 214, 231, 243, 321, 324, 341, 342, 412, 421, 423, 431.

EXAMPLE 23. 123, 124, 134, 143, 321, 324, 341, 421, 423, 431.

EXAMPLE 24. 123, 134, 142, 214, 241, 321, 324, 412, 423, 431.

In Examples 25–37, Postulate A fails, while B, C, and D hold.

EXAMPLE 25. 123, 142, 234, 341.

EXAMPLE 26. 123, 142, 143, 213, 214, 243, 413, 423, 421.

EXAMPLE 27. 123, 124, 243, 341.

EXAMPLE 28. 123, 143, 324, 421.

EXAMPLE 29. 123, 143, 213, 243, 412, 413, 423.

EXAMPLE 30. 123, 124, 143, 213, 214, 243, 413, 421, 423.

EXAMPLE 31. 123, 214, 341, 423.

EXAMPLE 32. 123, 142, 314, 412, 423.

EXAMPLE 33. 123, 142, 143, 324.

EXAMPLE 34. 123, 314, 412, 423.

EXAMPLE 35. 123, 124, 143, 243, 412, 423.

EXAMPLE 36. 123, 124, 143, 243, 423.

EXAMPLE 37. 123, 143, 214, 243, 412, 423.

In Example 38, Postulates A and C fail, while B and D hold.

EXAMPLE 38. 123, 143, 213, 214, 243, 412, 413, 421, 423.

In Examples 39 and 40, Postulates B and C fail, while A and D hold.

EXAMPLE 39. 123, 142, 214, 241, 321, 324, 412, 423.

EXAMPLE 40. 123, 143, 243, 321, 324, 341, 342, 423.

The properties possessed by these 44 systems are conveniently exhibited in Table II, in which a dot (.) indicates that a postulate is satisfied, while a cross (×) indicates that it is not satisfied.

TABLE II. EXAMPLES OF PSEUDO-BETWEENNESS

Ex.	A B C D	1 2 3 4 5 6 7 8	Lemma in which used.
A	×	.	A'
B	×	.	B'
C	×	.	C'
D	×	.	D'
1	.	×	1'a, 4'a, 5'b.
2	.	×	1'b, 2'b, 3'a.
3	.	×	2'a, 5'a, 6'a, 7'a, 8'a.
4	.	×	6'b, 8'c.
5	.	×	8'b.
6	×	.	1'c.
7	×	.	2'c, 7'b, 8'f.
8	×	.	2'd, 3'b.
9	×	.	3'c, 6'd, 8'e.
10	×	.	4'b, 5'c.
11	×	.	6'c, 7'c, 8'd.
12	.	×	1'd.
13	.	×	2'e.
14	.	×	3'd.
15	.	×	3'e.
16	.	×	4'c, 7'd.
17	.	×	4'd.
18	.	×	5'd, 7'e.
19	.	×	6'e, 7'f, 8'g.
20	.	×	7'g, 8'h.
21	.	×	5'e.
22	.	×	8'i.
23	.	×	8'j.
24	.	×	6'f.
25	×	.	1'e.
26	×	.	2'f.
27	×	.	3'f.
28	×	.	4'e.
29	×	.	4'f.
30	×	.	7'h.
31	×	.	6'h, 7'i, 8'o.
32	×	.	6'i.
33	×	.	5'f.
34	×	.	8'l.
35	×	.	8'm.
36	×	.	8'n.
37	×	.	4'g.
38	×	×	4'h.
39	×	×	6'g.
40	×	×	8'k.

By inspection of these 44 examples, we have at once 68 lemmas on non-deducibility, as exhibited in Table III.

TABLE III. LEMMAS ON NON-DEDUCIBILITY

Lemma	Postulate	is not deducible from postulates	Proof by example
A'	A	B C D 1 2 3 4 5 6 7 8	A
B'	B	A C D 1 2 3 4 5 6 7 8	B
C'	C	A B D 1 2 3 4 5 6 7 8	C
D'	D	A B C 1 2 3 4 5 6 7 8	D
1'a	1	A B C D 2 3 4 5 6 7 8	1
1'b	1	A B C D 4 5 6 7 8	2
1'c	1	A C D 2 3 4 5 6 7 8	6
1'd	1	A B D 2 3 4 5 6 7 8	12
1'e	1	B C D 2 3 4 5 6 7 8	25
2'a	2	A B C D 1 3 4 5 6 7 8	3
2'b	2	A B C D 4 5 6 7 8	2
2'c	2	A C D 1 3 4 5 6 7 8	7
2'd	2	A C D 1 4 5 6 7 8	8
2'e	2	A B D 1 3 4 5 6 7 8	13
2'f	2	B C D 1 3 4 5 6 7 8	26
3'a	3	A B C D 4 5 6 7 8	2
3'b	3	A C D 1 4 5 6 7 8	8
3'c	3	A C D 2 4 5 7	9
3'd	3	A B D 1 4 5 6 7 8	14
3'e	3	A B D 2 4 5 6 7 8	15
3'f	3	B C D 1 2 4 5 6 7 8	27
4'a	4	A B C D 2 3 4 5 6 7 8	1
4'b	4	A B C D 1 2 3 4 5 6 7 8	10
4'c	4	A B D 1 3 4 5 6 7 8	16
4'd	4	A B D 5 6 7 8	17
4'e	4	B C D 1 2 3 4 5 6 7 8	28
4'f	4	B C D 1 3 4 5 6 7 8	29
4'g	4	B C D 3 4 5 6 7 8	37
4'h	4	B D 1 3 4 5 6 7 8	38
5'a	5	A B C D 1 3 4 5 6 7 8	3
5'b	5	A B C D 2 3 4 5 6 7 8	1
5'c	5	A C D 1 2 3 4 5 6 7 8	10
5'd	5	A B D 1 3 4 5 6 7 8	18
5'e	5	A B D 4 5 6 7 8	21
5'f	5	B C D 1 2 3 4 5 6 7 8	33
6'a	6	A B C D 1 3 4 5 7	3
6'b	6	A B C D 4 5 7	4
6'c	6	A C D 1 2 3 4 5 7	11
6'd	6	A C D 2 4 5 7	9
6'e	6	A B D 1 2 3 4 5 7 8	19
6'f	6	A B D 4 5 7 8	24
6'g	6	A D 2 4 5 7 8	39
6'h	6	B C D 1 2 3 4 5 7 8	31
6'i	6	B C D 2 3 4 5 7 8	32
7'a	7	A B C D 1 3 4 5 6 8	3
7'b	7	A C D 1 3 4 5 6 8	7
7'c	7	A C D 1 2 3 4 5 6 8	11
7'd	7	A B D 1 3 4 5 6 8	16
7'e	7	A B D 1 3 4 5 6 8	18
7'f	7	A B D 1 2 3 4 5 6 8	19
7'g	7	A B D 3 4 5 6 8	20
7'h	7	B C D 1 3 4 5 6 8	30
7'i	7	B C D 1 2 3 4 5 6 8	31

TABLE III—Continued.

Lemma	Postulate	is not deducible from postulates										Proof by example
8'a	8	A	B	C	D	1		3	4			3
8'b	8	A	B	C	D					6	7	5
8'c	8	A	B	C	D				4	5	7	4
8'd	8	A		C	D	1	2	3	4	5		11
8'e	8	A		C	D		2		4	5	7	9
8'f	8	A		C	D	1		3			6	7
8'g	8	A	B		D	1	2	3	4	5		19
8'h	8	A	B		D			3	4	5	6	20
8'i	8	A	B		D				4	5	6	22
8'j	8	A	B		D	1		3	4		6	23
8'k	8	A			D	1		3		5	6	40
8'l	8		B	C	D		2	3	4	5	7	34
8'm	8		B	C	D			3	4	5	6	35
8'n	8		B	C	D	1		3	4	5	6	36
8'o	8		B	C	D	1	2	3	4	5		31

By comparing these lemmas with the theorems in § 2, we can now establish the following theorem of non-deducibility:

THEOREM. *No one of the twelve postulates of our basic list is deducible from any others of the list, except in the cases covered by our 71 theorems.*

For example, consider the case of Postulate 3. By Lemmas 3'a–3'f, we see that Postulate 3 can certainly not be proved without the use of at least one postulate from each of the following groups: 1, 2; B, 2; B, 1, 6, 8; C, 2; C, 1; A; hence the following combinations are the only ones which need to be investigated: A, 1, 2; A, B, C, 1; A, B, C, 2; A, C, 2, 6; A, C, 2, 8. But by reference to Theorems 3a–3e we see that each one of these combinations is in fact sufficient to prove Postulate 3.

The truth of the theorem for each of the other cases is established in a similar way.

4. ELEVEN SETS OF INDEPENDENT POSTULATES

We are now in position to select from our basic list of twelve postulates, several smaller lists which are free from redundancies.

An examination of our results in regard to deducibility shows that this selection can be made in precisely eleven ways; that is, there are precisely eleven sets of independent postulates which can be selected from our basic list.

The eleven sets are as follows:

- (1) A, B, C, D, 1, 2.
- (2) A, B, C, D, 1, 5.
- (3) A, B, C, D, 1, 6.
- (4) A, B, C, D, 1, 7.
- (5) A, B, C, D, 1, 8.
- (6) A, B, C, D, 2, 4.
- (7) A, B, C, D, 2, 5.
- (8) A, B, C, D, 3, 5.
- (9) A, B, C, D, 3, 4, 6.
- (10) A, B, C, D, 3, 4, 7.
- (11) A, B, C, D, 3, 4, 8.

That the postulates of each set are independent of one another is proved by the existence of Examples A-D, 1, 2, 3 above. (See Lemmas A-D, 1'a, 2'a, 3'a, 4'a, 5'a, 6'a, 7'a, 8'a, 1'b, 2'b, and 5'b.)

That the postulates of each set are sufficient to establish the entire list of twelve postulates is proved by our theorems on deducibility; in fact, in several cases the missing postulates can be deduced from the given postulates in more than one way. Table I, at the end of § 2, will show clearly all the possible ways in which the missing postulates in each set can be deduced from the given postulates of that set.

It will be noticed that certain theorems are not directly available in any of the eleven sets, since no one of the sets contains explicitly the postulates used in the proof of these theorems.

A comparison of the merits of the eleven sets of postulates by the aid of Table I, while perhaps not convincing in the present state of our knowledge of the standards to which such sets of postulates should conform, would at any rate be of some interest.

For example, if our aim is to find the set which shall be the most condensed, and from which the remaining postulates can be most readily deduced, we should select Set 1. If, on the other hand, our aim is to analyze the postulates down to their lowest terms, that is, to find a set from which the necessary deductions can just barely be made, we should then probably select Set 10. It is quite possible, of course, that some other considerations (not now clear) might lead us to select some other of the eleven sets as preferable for some purpose then in view.

In any case, it is satisfactory to know that these eleven sets are the only sets of independent postulates which can be selected from the basic list of twelve postulates from which we started.

Moreover, this basic list of twelve postulates must always occupy a central place in any theory of betweenness. For, as we have already pointed out, this set contains all the general laws concerning the betweenness relations among three or four elements; and even if further propositions concerning five or more distinct elements should be added to the list, no one of the basic list of twelve could thereby be made redundant. To prove this fact, we have merely to notice that the system exhibited in each of the examples used above in proving independence contains at most four elements, and would therefore satisfy vacuously any proposition involving five or more distinct elements.

## 5. DEFINITION OF BETWEENNESS

The following definition of betweenness may now be formulated:

DEFINITION. Any system  $(K, R)$  in which the class  $K$  and the triadic relation  $R$  are found to possess all the properties demanded by any one of

our sets of independent postulates (see § 4) may be called an *ordered class* or *series*, and the relation  $R$  itself may then be called the relation of *betweenness*.

The most familiar example of such an ordered class or series is the system  $(K, R)$  in which  $K$  is the class of points on a line, and  $AXB$  means that the point  $X$  belongs to the interior of the segment  $AB$ .

Another example is the system  $(K, R)$  in which  $K$  is the class of natural numbers and  $AXB$  means that the number  $X$  is larger than the smaller of the two numbers  $A$  and  $B$ , and smaller than the larger one.

In each of these examples we say that  $X$  is "between"  $A$  and  $B$ .

The relation between the theory of betweenness and the theory of serial order may be expressed as follows.

Let  $A$  and  $B$  be any two distinct elements of a "betweenness" system, and let  $X$  and  $Y$  be any other distinct elements of the system. Then we say that  $X$  *precedes*  $Y$ , in the order  $AB$ , if any one of the following conditions is true: (1)  $XAB$  and either  $XYA$  or  $Y = A$  or  $AYB$  or  $Y = B$  or  $ABY$ ; (2)  $X = A$  and either  $AYB$  or  $Y = B$  or  $ABY$ ; (3)  $AXB$  and either  $XYB$  or  $Y = B$  or  $ABY$ ; (4)  $X = B$  and  $ABY$ ; (5)  $ABX$  and  $BXY$ . From this definition, and the properties of betweenness, it is easy to derive the usual properties of the dyadic relation of serial order, always, however, with respect to the fixed base  $AB$ .

On the relation between the theory of betweenness and the theory of cyclic order, see a paper by E. V. Huntington: *A set of independent postulates for cyclic order*, Proceedings of the National Academy of Sciences, vol. 2 (1916), p. 630.

#### APPENDIX

*Remark Concerning Postulate A.*—In regard to the general laws of betweenness concerning four elements  $A, B, X, Y$  on a line, if we agree to read always in the direction from  $A$  towards  $B$ , the total number of these general laws appears at first sight to be twenty-four, which group themselves into nine groups, as follows:

$-X-----A-----B-----Y-----$		$-Y-----A-----B-X-----$	
1.	$XAB \cdot ABY \cdot \supset \cdot XAY$	1a.	$YAB \cdot ABX \cdot \supset \cdot YAX$
1c.	$XAB \cdot ABY \cdot \supset \cdot XBY$	1b.	$YAB \cdot ABX \cdot \supset \cdot YBX$
$-X-----A-Y-----B-----$		$-Y-----A-X-----B-----$	
2.	$XAB \cdot AYB \cdot \supset \cdot XAY$	2a.	$YAB \cdot AXB \cdot \supset \cdot YAX$
3.	$XAB \cdot AYB \cdot \supset \cdot XYB$	3a.	$YAB \cdot AXB \cdot \supset \cdot YXB$
$-----A-X-----B-Y-----$		$-----A-Y-----B-X-----$	
2c.	$AXB \cdot ABY \cdot \supset \cdot XBY$	2b.	$AYB \cdot ABX \cdot \supset \cdot YBX$
3c.	$AXB \cdot ABY \cdot \supset \cdot AXY$	3b.	$AYB \cdot ABX \cdot \supset \cdot AYX$



4. $AXB \cdot AYB \cdot \supset \cdot AXY \sim AYY$	$\text{-----} A \overset{X}{\text{---}} \overset{Y}{\text{---}} B \text{-----}$
4b. $AXB \cdot AYB \cdot \supset \cdot XYB \sim YXB$	
5. $AXB \cdot AYB \cdot \supset \cdot AXY \sim YXB$	5a. $AXB \cdot AYB \cdot \supset \cdot AYY \sim XYB$
6. $XAB \cdot YAB \cdot \supset \cdot XYB \sim YXB$	$\overset{X}{\text{---}} \overset{Y}{\text{---}} A \text{-----} B \text{-----}$
7. $XAB \cdot YAB \cdot \supset \cdot XYA \sim YXA$	
8. $XAB \cdot YAB \cdot \supset \cdot XYA \sim YXB$	8a. $XAB \cdot YAB \cdot \supset \cdot YXA \sim XYB$
6b. $ABX \cdot ABY \cdot \supset \cdot AXY \sim AYY$	$\text{-----} A \text{-----} B \overset{X}{\text{---}} \overset{Y}{\text{---}}$
7b. $ABX \cdot ABY \cdot \supset \cdot BXY \sim BYX$	
8b. $ABX \cdot ABY \cdot \supset \cdot AXY \sim BYX$	8c. $ABX \cdot ABY \cdot \supset \cdot AYY \sim BXY$

But each of the postulates in the second column is immediately obtainable from the postulate standing opposite it in the first column, without the use of any other postulate, so that the list of 24 is at once reducible to 15.

Furthermore, any two of the 24 postulates which bear the same number are deducible from each other by the aid of Postulate A alone. Hence the list of 24 reduces to 8, which may be selected in various ways; all these selections are equivalent in view of Postulate A; the Postulates 1-8 of the text represent one such selection.

On the other hand, if we agree to read either forward or backward along the line, the list of 24 would have to be greatly enlarged, so as to include, for example, such postulates as  $XAB \cdot YBA \cdot \supset \cdot YAX$ . All such postulates are immediately deducible from Postulates 1-8 by the aid of Postulate A, and are not here considered. It should be noted, however, that if it were desired to give a complete discussion of what could be proved without the aid of Postulate A, it would be necessary to consider the whole of the enlarged list, and also to modify slightly the wording of Postulate C.

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